Dynamic Response of Elastically Supported Circular Plates to a General Surface Load

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An analytical study is presented for the general dynamic response of elastically supported circular plates with an initial tension subjected to an arbitrary surface load $p(r, \theta, t)$. The plates are considered to be supported by a Winkler elastic foundation and elastically constrained against rotation at the outer edge. General natural frequency equations are presented with numerical values included for the first four modes of vibration plotted as functions of initial tension and elastic edge constraints ranging from clamped to simply supported edges. The general solution for forced response is given in an integral form based on Fourier-Bessel techniques with several useful examples included. Static buckling loads are also included for a range of limiting cases.

Nomenclature

a	= plate radius at outer edge
A_{mn}, B_{mn} C_{mn}, D_{mn}	= amplitude parameter functions
$A(m, n, t)$	= surface load integral defined by Eq. (13)
$B(m, n, t)$	= surface load integral defined by Eq. (17)
$A_{mn}{}^*, B_{mn}{}^*$	= particular solutions of amplitude functions dependent on $p(r, \theta, t)$
D	= flexural rigidity of plate
E	= modulus of elasticity
h	= plate thickness
I_n	= modified Bessel functions of order n
J_n	= Bessel functions of order n
k_1	= Winkler elastic foundation parameter
k_2	= rotational spring constant at outer edge of plate
m	= number of nodal circles
n	= number of nodal diameters
$p(r, \theta, t)$	= general surface loading function
Q_{mn}	= orthogonality constant
r, θ	= polar coordinates
$R_{mn}(r)$	= radial mode shape parameter defined by Eq. (10)
t	= time
T	= plate tension
$\bar{T}$	= magnitude of buckling load for first mode.
w	= transverse plate deflection
$\bar{\alpha}_{00}$	= eigenvalue for first buckling mode
α_{mn}, γ_{mn}	= eigenvalues associated with J_n and I_n, respectively
β	= edge rotational constraint parameter equal to $\nu + (k_2 a/D)$
δ	= Dirac delta symbol
ν	= Poisson's ratio
ρ	= mass density
$\dot{\phi}$	= constant angular velocity of circularly moving load
Ω_{mn}	= circular frequency parameter
ω	= circular frequency of applied surface load

Introduction

THE study of vibrations of circular plates has a long history dating back to 1829 when Poisson[1] studied radially symmetrical free vibrations, followed by a general solution for free vibrations given by Kirchoff[2] in 1850. Since that time many authors have considered various special cases of free and forced vibrations which usually involved either restricting the work to axisymmetric vibrations or considering only clamped and/or simply supported edges.

Some of the more recent studies dealing with nonaxisymmetric vibrations include work by Reismann,[3] who determined the forced response of a clamped circular plate due to an arbitrarily placed harmonic concentrated force, and Wah,[4] who presented natural frequency data for simply supported and clamped circular plates under initial tension. In addition, McLeod and Bishop[5] presented solutions for a variety of special cases of forced vibrations of circular plates with simply supported, free, clamped and sliding circular boundaries. Later, Anderson[6] presented a rather general finite integral transform method for studying the forced response of circular plates although the effects of initial tension and elastic constraints were not discussed.

It is the purpose of this analysis to present a general integral solution based on Fourier-Bessel techniques for the free and forced response of elastically supported circular plates subjected to an arbitrary surface load $p(r, \theta, t)$, including the effects of initial tension. The plate is considered to be supported by a Winkler elastic foundation and elastically constrained against rotation at the outer edge. The general solution is based on orthogonality conditions similar to those used by Weiner[7] in analyzing forced axisymmetric vibrations with elastic edge constraints and by Schlack and Kessel[8] in studying the nonaxisymmetric forced response of circular plates attached to an Euler reference frame subjected to combined spin and precession. The integral solution presented may be readily integrated to determine the plate response for any applied surface loading $p(r, \theta, t)$.

General Analysis

Equations of Motion

The homogeneous, isotropic, circular plate of radius a shown in Fig. 1 is subjected to a uniform radial tension T and is supported by a Winkler foundation of effective stiffness k_1. In addition, the edges of the plate are elastically constrained against rotation represented by the torsional spring of stiffness k_2. The governing partial differential equation of motion for the plate subjected to a general transverse surface load $p(r, \theta, t)$ is given according to elementary thin plate theory by

$$D\nabla^2\nabla^2 w - T\nabla^2 w + \rho h(\partial^2 w/\partial t^2) + k_1 w = p(r, \theta, t) \qquad (1)$$

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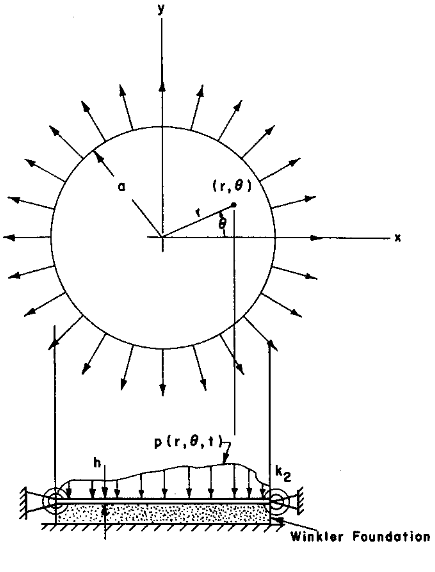

Fig. 1 Plate geometry and general loading $p(r,\theta,t)$.

The boundary conditions at the edge $r = a$ are given by

$$\begin{aligned} w &= 0 \\ -D[\partial^2 w/\partial r^2 + (\nu/r)\partial w/\partial r + (\nu/r^2)\partial^2 w/\partial\theta^2] &= k_2 \partial w/\partial r \end{aligned} \tag{2}$$

Thus, simply supported edges are the result of taking $k_2 = 0$ in Eq. (2) and clamped edges are realized by letting $k_2 \to \infty$ in the subsequent analysis.

Homogeneous Solution

The homogeneous solution of Eq. (1) according to the method of separation of variables is of the form

$$w = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[\{A_{mn}\sin n\theta + B_{mn}\cos n\theta\}\, J_n\left(\alpha_{mn}\frac{r}{a}\right) + \{C_{mn}\sin n\theta + D_{mn}\cos n\theta\}\, I_n\left(\gamma_{mn}\frac{r}{a}\right)\right] e^{i\Omega_{mn}t} \tag{3}$$

where A_{mn}, B_{mn}, C_{mn}, and D_{mn} are constants, J_n and I_n are Bessel and modified Bessel functions of order n of the first kind, respectively, and α_{mn} and γ_{mn} are related by the equation

$$\gamma_{mn}^2 - \alpha_{mn}^2 = Ta^2/D \tag{4}$$

Furthermore, the circular frequencies of the plate are given by

$$\Omega_{mn}^2 = (D/\rho h)(\alpha_{mn}/a)^4 + (T/\rho h)(\alpha_{mn}/a)^2 + k_1/\rho h \tag{5}$$

The boundary condition given by Eq. (2) requiring zero deflection at the edge of the plate leads to the relationship

$$A_{mn}/C_{mn} = B_{mn}/D_{mn} = -I_n(\gamma_{mn})/J_n(\alpha_{mn}) \tag{6}$$

The elastic rotational constraint at the boundary which is expressed by Eq. (2) requires for nontrivial solutions that the roots α_{mn} and γ_{mn} satisfy the equation

$$\alpha_{mn}\frac{J_{n+1}(\alpha_{mn})}{J_n(\alpha_{mn})} + \gamma_{mn}\frac{I_{n+1}(\gamma_{mn})}{I_n(\gamma_{mn})} = \frac{\alpha_{mn}^2 + \gamma_{mn}^2}{1 - \nu - (k_2 a/D)} \tag{7}$$

Thus, the solution to the homogeneous equation can be expressed as

$$w = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}[A_{mn}\sin n\theta + B_{mn}\cos n\theta]\, R_{mn}(r)\, e^{i\Omega_{mn}t} \tag{8}$$

where

$$R_{mn}(r) = J_n\left(\alpha_{mn}\frac{r}{a}\right) - \frac{J_n(\alpha_{mn})}{I_n(\gamma_{mn})} I_n\left(\gamma_{mn}\frac{r}{a}\right) \tag{9}$$

The circular frequency Ω_{mn} may be determined by Eq. (5) after the roots α_{mn} and γ_{mn} are obtained from the simultaneous solution of Eqs. (4) and (7).

Forced Response

Using the characteristic functions from Eq. (8), an appropriate solution for the forced response may be written in the form

$$w = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}[A_{mn}\sin n\theta + B_{mn}\cos n\theta]\, R_{mn}(r) \tag{10}$$

where A_{mn} and B_{mn} are now considered to be functions of time. Substitution of Eq. (10) into Eq. (1) may be shown to yield the relationship

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}[\{\ddot{A}_{mn} + \Omega_{mn}^2 A_{mn}\}\sin n\theta + \{\ddot{B}_{mn} + \Omega_{mn}^2 B_{mn}\}\cos n\theta]\, R_{mn}(r) = \frac{p(r,\theta,t)}{\rho h} \tag{11}$$

By multiplying both sides of Eq. (11) by $R_{ps}(r)\sin s\theta$ and applying the orthogonality properties discussed in the Appendix we obtain

$$\ddot{A}_{mn} + \Omega_{mn}^2 A_{mn} = A(m,n,t) \tag{12}$$

where

$$A(m,n,t) = \frac{1}{\rho h Q_{mn}}\int_0^{2\pi}\int_0^a p(r,\theta,t)\{R_{mn}(r)\sin n\theta\}\, r\, dr\, d\theta \tag{13}$$

and

$$Q_{mn} = \pi\int_0^a R_{mn}^2(r) r\, dr \tag{14}$$

Upon integrating Eq. (14) it may be shown that

$$Q_{mn} = \pi a^2\left[J_n^2(\alpha_{mn}) - \left(1 - \frac{\alpha_{mn}^2}{\gamma_{mn}^2}\right)\left\{\frac{1}{2}J_{n+1}^2(\alpha_{mn}) + J_{n+1}(\alpha_{mn})J_n'(\alpha_{mn})\right\}\right] - \frac{\pi a^2}{1-\nu-(k_2 a/D)}\left[\left\{2 + \frac{(\alpha_{mn}^2+\gamma_{mn}^2)^2}{2\gamma_{mn}^2}\right\} \times J_n^2(\alpha_{mn}) + \alpha_{mn}\frac{\alpha_{mn}^2+\gamma_{mn}^2}{\gamma_{mn}^2} J_n(\alpha_{mn})J_n'(\alpha_{mn})\right] \tag{15}$$

Clearly, the second set of terms is zero for a clamped outer boundary as seen by letting $k_2 \to \infty$.

Similarly we may write that

$$\ddot{B}_{mn} + \Omega_{mn}^2 B_{mn} = B(m,n,t) \tag{16}$$

where

$$B(m,n,t) = \frac{1}{\rho h Q_{mn}}\int_0^{2\pi}\int_0^a p(r,\theta,t)\{R_{mn}(r)\cos n\theta\}\, r\, dr\, d\theta \tag{17}$$

The general solutions for the functions A_{mn} and B_{mn} for an arbitrary loading $p(r,\theta,t)$ are given by

$$A_{mn} = a_{mn}\sin\Omega_{mn}t + c_{mn}\cos\Omega_{mn}t + A_{mn}^* \tag{18}$$

and

$$B_{mn} = b_{mn}\sin\Omega_{mn}t + d_{mn}\cos\Omega_{mn}t + B_{mn}^* \tag{19}$$

where

$$A_{mn}^* = \frac{1}{\Omega_{mn}}\int_0^t A(m,n,\tau)\sin\Omega_{mn}(t-\tau)\, d\tau \tag{20}$$

$$B_{mn}^* = \frac{1}{\Omega_{mn}}\int_0^t B(m,n,\tau)\sin\Omega_{mn}(t-\tau)\, d\tau \tag{21}$$

and the constants a_{mn}, b_{mn}, c_{mn}, d_{mn} are zero for homogeneous initial conditions.

Thus, Eqs. (18) and (19) in conjunction with Eq. (10) complete the general solution for the response of elastically supported circular plates subjected to an arbitrary surface load $p(r,\theta,t)$.

Eigenvalues and Natural Frequencies

In order to apply the equations of the last section to the solution of forced vibration problems it is necessary to know the roots α_{mn} and γ_{mn} and the characteristic frequency Ω_{mn} for use in the series expansion represented by Eq. (10). However, before proceeding it is convenient to represent the tension T in nondimensional form by dividing it by the magnitude of the lowest buckling load associated with each edge constraint constant k_2, with $k_1 = 0$.

Buckling loads for $m = n = 0$ can be determined from Eq. (5) by setting $\Omega_{00} = 0$ and solving for $\bar{T}$ in terms of $\bar{\alpha}_{00}$ for $k_1 = 0$ yielding

$$\bar{T} = |-D\bar{\alpha}_{00}{}^2/a^2| = D\bar{\alpha}_{00}{}^2/a^2 \qquad (22)$$

Thus, Eqs. (4) and (5) can be rewritten in nondimensional form as

$$\gamma_{mn}{}^2 - \alpha_{mn}{}^2 = \bar{\alpha}_{00}{}^2 T/\bar{T} \qquad (23)$$

and

$$\frac{\rho h a^4}{D}\Omega_{mn}{}^2 = \alpha_{mn}{}^4\left(1 + \frac{\bar{\alpha}_{00}{}^2}{\alpha_{mn}{}^2}\frac{T}{\bar{T}}\right) + \frac{a^4 k_1}{D} \qquad (24)$$

Values of $\bar{\alpha}_{00}{}^2$ are given in Table 1 for the range of values of k_2 considered in Eq. (7), letting $\beta = \nu + k_2 a/D$.

Table 1 Buckling load parameters for $m = n = k_1 = 0$

β	0	0.3	1.0	10	100	∞
$\bar{\alpha}_{00}{}^2$	3.40	4.20	5.76	12.1	14.4	14.7

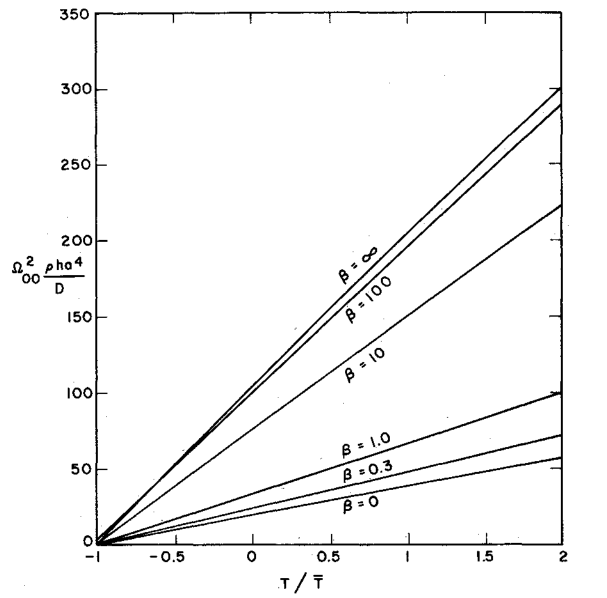

Fig. 2 Natural circular frequencies for $m = 0$ (nodal circle) and $n = 0$ (nodal diameter).

The results of simultaneous solution of Eqs. (23) and (7) for the roots α_{mn} for a wide range of initial tensions and edge rotational spring constants k_2 are given in Table 2 for $m, n = 0, 1$. Note that the roots γ_{mn} can then be readily determined from Eq. (23) for each value of k_2 considered.

In addition, the nondimensional circular frequencies are plotted in Figs. 2–5 for $m, n = 0, 1$ vs $T/\bar{T}$ for a wide range of values for k_1 taken equal to zero. Note however that all tensions are nondimensionalized with respect to the lowest buckling load for each value of k_2 according to the values given in Table 1. If the Winkler elastic foundation constant k_1 is different than zero it is necessary according to Eq. (24) to simply add $a^4 k_1/D$ to the corresponding nondimensional frequency parameter represented in Figs. 2–5 to take this factor into account.

Figures 2–5 shows that in general the circular frequency increases as the tension is increased and as the elastic edge constraint constant k_2 increases. Although the curves can be shown to be slightly nonlinear for the range of variables studied, they can be approximated by straight lines for design purposes thus making interpolation easy.

Table 2 Roots α_{mn} of Eqs. (23) and (7), where m and n are the number of nodal circles and diameters, respectively

$\nu + \frac{k_2 a}{D}$		0		0.3		1.0		10		100		∞	
$T/\bar{T}$	m \ n	0	1	0	1	0	1	0	1	0	1	0	1
2.0	0	2.240	3.707	2.307	3.753	2.405	3.832	2.669	4.140	2.814	4.351	2.840	4.391
	1	5.429	6.943	5.460	6.967	5.520	7.016	5.836	7.318	6.108	7.630	6.165	7.699
1.5	0	2.220	3.700	2.295	3.748	2.405	3.832	2.707	4.169	2.861	4.390	2.889	4.431
	1	5.427	6.942	5.458	6.966	5.520	7.016	5.854	7.331	6.136	7.651	6.194	7.721
1.0	0	2.194	3.693	2.278	3.742	2.405	3.832	2.760	4.203	2.927	4.436	2.955	4.478
	1	5.424	6.941	5.456	6.965	5.520	7.016	5.876	7.345	6.168	7.674	6.226	7.744
0.50	0	2.159	3.684	2.255	3.736	2.405	3.832	2.835	4.246	3.020	4.494	3.049	4.537
	1	5.422	6.939	5.454	6.963	5.520	7.016	5.900	7.360	6.205	7.699	6.263	7.770
0.25	0	2.136	3.680	2.240	3.732	2.405	3.832	2.887	4.271	3.084	4.528	3.114	4.571
	1	5.420	6.939	5.453	6.963	5.520	7.016	5.913	7.368	6.225	7.713	6.284	7.784
0	0	2.108	3.674	2.222	3.728	2.405	3.832	2.953	4.300	3.165	4.567	3.196	4.610
	1	5.419	6.938	5.452	6.963	5.520	7.016	5.928	7.377	6.247	7.727	6.306	7.799
−0.25	0	2.073	3.669	2.198	3.728	2.405	3.832	3.038	4.333	3.269	4.611	3.302	4.655
	1	5.417	6.937	5.450	6.962	5.520	7.016	5.944	7.386	6.271	7.742	6.331	7.814
−0.50	0	2.025	3.663	2.167	3.719	2.405	3.832	3.150	4.371	3.404	4.661	3.438	4.706
	1	5.416	6.937	5.449	6.962	5.520	7.016	5.961	7.396	6.297	7.758	6.357	7.830
−1.0	0	1.841	3.650	2.049	3.709	2.405	3.832	3.480	4.466	3.794	4.787	3.831	4.833
	1	5.413	6.935	5.446	6.960	5.520	7.016	6.000	7.417	6.355	7.793	6.417	7.865

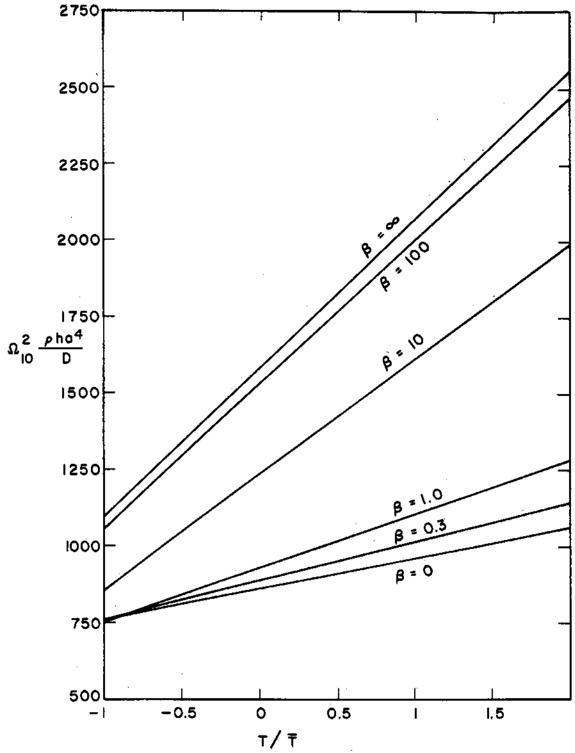

Fig. 3 Natural circular frequencies for $m = 0$ (nodal circle) and $n = 1$ (nodal diameter).

Forced Response Examples

The general forced response of elastically constrained circular plates due to an arbitrary surface load is given by Eq. (10) where the functions A_{mn} and B_{mn} are expressed by Eqs. (18–21). In order to further demonstrate the method, in addition to presenting useful design results, the particular solutions of Eqs. (18) and (19) are presented in the following examples for a limited number of representative cases. For convenience homogeneous initial conditions are assumed for these examples.

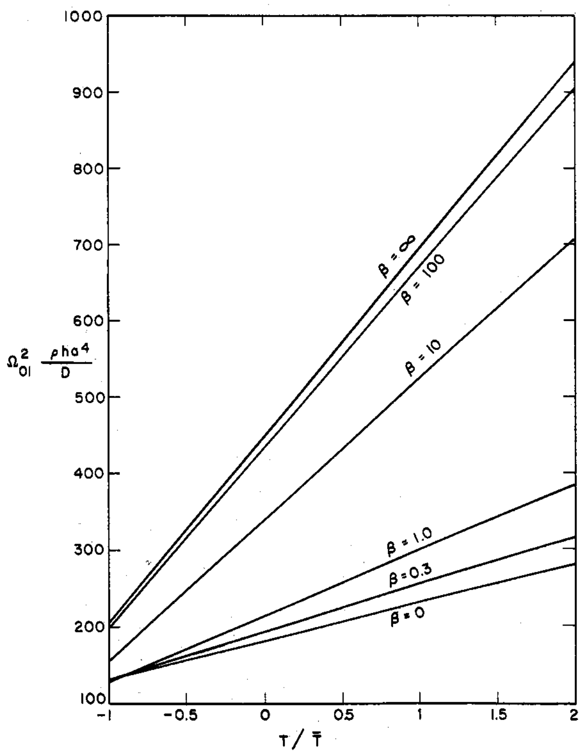

Fig. 4 Natural circular frequencies for $m = 1$ (nodal circle) and $n = 0$ (nodal diameter).

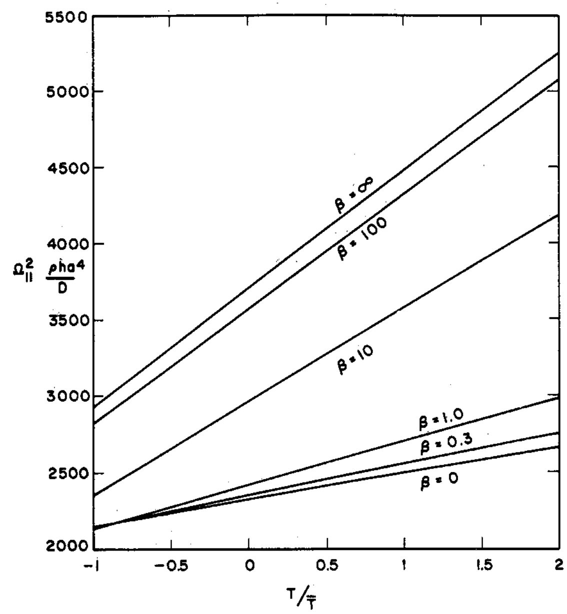

Fig. 5 Natural circular frequencies for $m = 1$ (nodal circle) and $n = 1$ (nodal diameter).

a. Concentrated Harmonic Force

The loading function for a concentrated harmonic force of magnitude $P \cos \omega t$ applied at $r = r_0$ and $\theta = 0$ (arbitrary) is represented by

$$p(r, \theta, t) = P \cos \omega t[\delta(r - r_0)\delta(\theta)] \tag{25}$$

where δ is the Dirac-delta function. Substituting Eq. (25) into Eqs. (13) and (17) and integrating, we have in accordance with Eqs. (20) and (21) that the general solutions for the amplitude functions A_{mn} and B_{mn} are

$$A_{mn}^* = 0$$

$$B_{mn}^* = \frac{P r_0 R_{mn}(r_0)}{\rho h Q_{mn}(\Omega_{mn}^2 - \omega^2)} (\cos \omega t - \cos \Omega_{mn} t) \tag{26}$$

where Q_{mn} is given by Eq. (15) for each pair of roots α_{mn}, γ_{mn} and $R_{mn}(r_0)$ is defined according to Eq. (10) as

$$R_{mn}(r_0) = J_n\left(\alpha_{mn} \frac{r_0}{a}\right) - \frac{J_n(\alpha_{mn})}{I_n(\gamma_{mn})} I_n\left(\gamma_{mn} \frac{r_0}{a}\right) \tag{27}$$

In Eq. (26) the term containing $\cos \Omega_{mn} t$ represents the effect of free vibration on the displacement and will soon be damped out in all practical systems. Thus they may be disregarded in order to determine the steady-state response.

Accordingly, the steady state response at any point (r, θ) at time t is given by

$$w = \frac{P r_0}{\rho h} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{R_{mn}(r_0)}{Q_{mn}(\Omega_{mn}^2 - \omega^2)} R_{mn}(r) \cos n\theta \cos \omega t \tag{28}$$

This solution is a generalization of the work presented in Refs. 3 and 5 for the response of elastically supported circular plates to a concentrated harmonic force $P \cos \omega t$, where herein the edge rotational spring constant k_2 influences Q_{mn}, Ω_{mn} and R_{mn} in Eq. (28). Also, the effects of an elastic Winkler foundation and initial prestressing are included.

b. Concentrated Harmonic Couple

The concentrated surface couple acting vectorially in the plane of the plate at $r = r_0$ and $\theta = 0$ (arbitrary) shown in Fig. 6 can

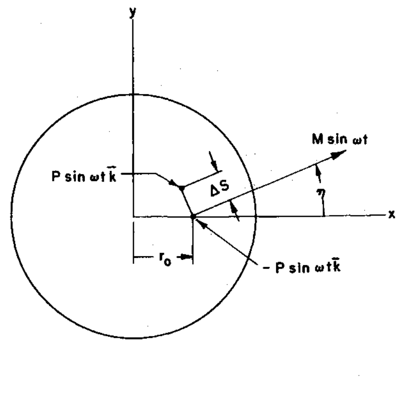

Fig. 6 Surface couple loading $M \sin \omega t$.

be conveniently represented by a pair of equal and opposite concentrated forces $P \sin \omega t$ acting normal to the plate a small distance ΔS apart as regularly done for doublet functions.

As a result, we may simply superimpose the responses for this doublet pair according to the results of the previous example and take the limit as

$$\Delta S \to 0 \quad \text{with} \quad \lim_{\Delta S \to 0} [P \Delta S] = M$$

to determine the steady-state response of a circular plate to a concentrated harmonic couple $M \sin \omega t$ which is given by

$$w = \frac{M r_0}{\rho h} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{R_{mn}(r_0)}{Q_{mn}} \left[\left\{ \frac{n}{r} R_{mn}(r) - \frac{\alpha_{mn}}{a} J_{n+1}\left(\alpha_{mn} \frac{r}{a}\right) - \frac{\gamma_{mn}}{a} \frac{J_n(\alpha_{mn})}{I_n(\gamma_{mn})} I_{n+1}\left(\gamma_{mn} \frac{r}{a}\right) \right\} \cos \eta \cos n\theta - \frac{n}{r_0} R_{mn}(r) \sin \eta \sin n\theta \right] \cos \omega t \quad (29)$$

c. Load Moving in a Circular Path

In order to demonstrate the use of this analysis for a moving load, consider a load $P \cos \omega t$ that travels at constant angular velocity $\dot{\phi}$ along a circular path of radius r_0 about the center of the plate. The loading function $p(r, \theta, t)$ for this case is written

$$p = P \sin \omega t [\delta(r - r_0) \delta(\theta - \dot{\phi} t)] \quad (30)$$

where P, ω, r_0 and $\dot{\phi}$ are constants.

Substituting Eq. (30) into Eqs. (13) and (17) are subsequently into Eqs. (20) and (21) we obtain the steady-state solutions for this case given by

$$A_{mn}^* = \frac{P r_0 R_{mn}(r_0)}{2 \rho h Q_{mn}} \left[\frac{\sin (n\dot{\phi} - \omega)t}{\Omega_{mn}^2 - (n\dot{\phi} - \omega)^2} + \frac{\sin (n\dot{\phi} + \omega)t}{\Omega_{mn}^2 - (n\dot{\phi} + \omega)^2} \right] \quad (31)$$

and

$$B_{mn}^* = \frac{P r_0 R_{mn}(r_0)}{2 \rho h Q_{mn}} \left[\frac{\cos (n\dot{\phi} - \omega)t}{\Omega_{mn}^2 - (n\dot{\phi} - \omega)^2} + \frac{\cos (n\dot{\phi} + \omega)t}{\Omega_{mn}^2 - (n\dot{\phi} + \omega)^2} \right] \quad (32)$$

Thus, the steady-state response is given according to Eq. (10) by

$$w = \frac{P r_0}{2 \rho h} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{R_{mn}(r_0)}{Q_{mn}} \left[\left\{ \frac{\sin (n\dot{\phi} - \omega)t}{\Omega_{mn}^2 - (n\dot{\phi} - \omega)^2} + \frac{\sin (n\dot{\phi} + \omega)t}{\Omega_{mn}^2 - (n\dot{\phi} + \omega)^2} \right\} \sin n\theta + \left\{ \frac{\cos (n\dot{\phi} - \omega)t}{\Omega_{mn}^2 - (n\dot{\phi} - \omega)^2} + \frac{\cos (n\dot{\phi} + \omega)t}{\Omega_{mn}^2 - (n\dot{\phi} + \omega)^2} \right\} \cos n\theta \right] R_{mn}(r) \quad (33)$$

It is particularly interesting to note that resonance occurs for this circularly moving load whenever $\Omega_{mn} = \pm n\dot{\phi} \pm \omega$ where $n = 0, 1, \ldots \infty$. Thus, resonance is greatly influenced by the rate $\dot{\phi}$ at which the load moves about the center of the plate and even for a constant load P resonance occurs for $\Omega_{mn} = n\dot{\phi}$.

d. Pulsating Uniform Surface Pressure

An important axisymmetric loading consists of a pulsating pressure $p \cos \omega t$ acting uniformly over a circular region $0 < r \leq r_0$. The corresponding load function is given by

$$p(r, \theta, t) = \begin{cases} p \cos \omega t; & 0 < r \leq r_0 \\ 0 \quad ; & r_0 < r < a \end{cases} \quad (34)$$

The steady-state solutions of Eqs. (18) and (19) since $n = 0$ for the axisymmetric case are given by

$$A_{m0}^* = 0$$

$$B_{m0}^* = \frac{2\pi p r_0 \cos \omega t}{\rho h Q_{m0} (\Omega_{m0}^2 - \omega^2)} \left[J_1\left(\alpha_{m0} \frac{r_0}{a}\right) - \frac{J_0(\alpha_{m0})}{I_0(\gamma_{m0})} I_1\left(\gamma_{m0} \frac{r_0}{a}\right) \right] \quad (35)$$

resulting in the steady-state response

$$w = \sum_{m=0}^{\infty} B_{m0}^* R_{m0}(r) \quad (36)$$

where $R_{m0}(r)$ is given by Eq. (10) for $n = 0$. Note that resonance occurs for each axisymmetric mode for $\omega = \Omega_{m0}$.

Closure

The analysis presented herein thus provides a unified approach for determining the free and forced response of elastically supported circular plates subjected to an arbitrary surface load $p(r, \theta, t)$. The general solution for forced response is presented in such a way that it may be readily applied to a wide variety of engineering design situations.

The dependence of the natural frequencies on the initial tension, the Winkler foundation stiffness and the edge rotational stiffness is shown in Figs. 2–5 for the complete spectrum of edge constraints from clamped to simply supported for the first four natural frequencies of the system. In addition, the appropriate set of equations for determining higher order natural frequencies are presented for the complete range of elastic support constants with the effects of initial tension or compression below the buckling load included.

Appendix

Referring to the table of formulas given by McLachlan,[9] the integral

$$\int_0^a \left\{ J_n\left(\alpha_{mn} \frac{r}{a}\right) - \frac{J_n(\alpha_{mn})}{I_n(\gamma_{mn})} I_n\left(\gamma_{mn} \frac{r}{a}\right) \right\} \left\{ J_n\left(\alpha_{sn} \frac{r}{a}\right) - \frac{J_n(\alpha_{sn})}{I_n(\gamma_{sn})} I_n\left(\gamma_{sn} \frac{r}{a}\right) \right\} r\, dr = \frac{a^2}{\alpha_{mn}^2 - \alpha_{sn}^2} \times$$
$$\{\alpha_{mn} J_{n+1}(\alpha_{mn}) J_n(\alpha_{sn}) - \alpha_{sn} J_n(\alpha_{mn}) J_{n+1}(\alpha_{sn})\} - \frac{J_n(\alpha_{mn})}{I_n(\gamma_{mn})} \times$$
$$\frac{a^2}{\gamma_{mn}^2 + \alpha_{sn}^2} \{\gamma_{mn} I_{n+1}(\gamma_{mn}) J_n(\alpha_{sn}) + \alpha_{sn} I_n(\gamma_{mn}) J_{n+1}(\alpha_{sn})\} -$$
$$\frac{J_n(\alpha_{sn})}{I_n(\gamma_{sn})} \frac{a^2}{\alpha_{mn}^2 + \gamma_{sn}^2} \{\alpha_{mn} J_{n+1}(\alpha_{mn}) I_n(\gamma_{sn}) + \gamma_{sn} J_n(\alpha_{mn}) I_{n+1}(\gamma_{sn})\} +$$
$$\frac{J_n(\alpha_{mn})}{I_n(\gamma_{mn})} \frac{J_n(\alpha_{sn})}{I_n(\gamma_{sn})} \frac{a^2}{\gamma_{mn}^2 - \gamma_{sn}^2} \times$$
$$\{\gamma_{mn} I_{n+1}(\gamma_{mn}) I_n(\gamma_{sn}) - \gamma_{sn} I_n(\gamma_{mn}) I_{n+1}(\gamma_{sn})\} \quad (A1)$$

for $s \neq m$. The sets of roots γ_{mn}, α_{mn} and γ_{sn}, α_{sn} must each satisfy Eq. (5) and as a result are related to each other by the condition

$$\gamma_{mn}^2 - \gamma_{sn}^2 = \alpha_{mn}^2 - \alpha_{sn}^2 \quad (A2)$$

In addition, the boundary conditions at the outer edge of the plate require that each set of roots satisfies the respective equation:

$$\alpha_{mn}\frac{J_{n+1}(\alpha_{mn})}{J_n(\alpha_{mn})}+\gamma_{mn}\frac{I_{n+1}(\gamma_{mn})}{I_n(\gamma_{mn})}=\frac{\alpha_{mn}{}^2+\gamma_{mn}{}^2}{1-v-(k_2a/D)} \tag{A3}$$

or

$$\alpha_{sn}\frac{J_{n+1}(\alpha_{sn})}{J_n(\alpha_{sn})}+\gamma_{sn}\frac{I_{n+1}(\gamma_{sn})}{I_n(\gamma_{sn})}=\frac{\alpha_{sn}{}^2+\gamma_{sn}{}^2}{1-v-(k_2a/D)} \tag{A4}$$

Substituting Eqs. (A2, A3, and A4) into Eq. (A1) it is readily verified that for $s \neq m$

$$\int_0^a \left\{J_n\left(\alpha_{mn}\frac{r}{a}\right)-\frac{J_n(\alpha_{mn})}{I_n(\gamma_{mn})}I_n\left(\gamma_{mn}\frac{r}{a}\right)\right\}\left\{J_n\left(\alpha_{sn}\frac{r}{a}\right)-\frac{J_n(\alpha_{sn})}{I_n(\gamma_{sn})}I_n\left(\gamma_{sn}\frac{r}{a}\right)\right\}r\,dr=0 \tag{A5}$$

for elastically supported outer edges at $r = a$ for any value of the rotational spring constant k_2.

References

[1] Poisson, S. D., "Sur le mouvement des corps elastique," Memoires de l'Academic Koyale des Sciences de l'Institut de France, Vol. 8, 1829, pp. 357–370.

[2] Kirchhoff, G., "Über das Gleichgewicht und Bewegung einer Elastischen Scheibe," *Journal für die reine und angenwandte Mathematik (Crelle)*, Vol. 40, 1850, p. 51.

[3] Reismann, H., "Forced Vibrations of a Circular Plate," *Journal of Applied Mechanics*, Vol. 26, 1959, pp. 526–527.

[4] Wah, T., "Vibration of Circular Plates," *Journal of the Acoustical Society of America*, Vol. 34, No. 3, 1962, pp. 275–281.

[5] McLeod, A. J. and Bishop, R. E. D., "The Forced Vibration of Circular Flat Plates," Mechanical Engineering Science-Monograph No. 1, 1965.

[6] Anderson, G., "On the Determination of Finite Integral Transforms for Forced Vibrations of Circular Plates," *Journal of Sound and Vibration*, Vol. 9, 1969, pp. 126–144.

[7] Weiner, R. S., "Forced Axisymmetric Motions of Circular Elastic Plates," *Journal of Applied Mechanics*, Vol. 4, 1965, pp. 893–898.

[8] Schlack, A. L. and Kessel, P. G., "Gyroscopically Induced Vibration of Circular Plates," *AIAA Journal*, Vol. 7, No. 10, Oct. 1969, pp. 1932–1937.

[9] McLachlan, N. W., *Bessel Functions for Engineers*, 2nd ed., Clarendon Press, Oxford, England, 1955.